Girraween High School

Year 12 Mathematics Extension 2

Task 1, December 2024

Question	Mark	MEX12-1	MEX12-4	MEX12-7	MEX12-8
1-5	/5	✓	✓	✓	✓
6	/11	✓	✓	✓	✓
7	/9	✓	✓	✓	✓
8	/12	✓	✓	✓	✓
9	/10	✓	✓	✓	✓
10	/11	✓	✓	✓	✓
11	/8	✓	✓	✓	✓
Total	/66	/66	/66	/66	/66

Year 12 Mathematics Extension 2 outcomes

A student:

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts

MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings

MEX12-3 uses vectors to model and solve problems in two and three dimensions

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

MEX12-5 applies techniques of integration to structured and unstructured problems

MEX12-6 uses mechanics to model and solve practical problems

MEX12-7 applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems

MEX12-8 communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

Year 12 Mathematics Extension 2
Task 1 – December 2024
Multiple Choice Answer Sheet

Student Number: _____

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

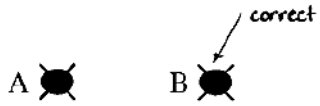
Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.



Select the alternative A, B, C or D that best answers the question.

1. A ○ B. ○ C. ○ D. ○
2. A ○ B. ○ C. ○ D. ○
3. A ○ B. ○ C. ○ D. ○
4. A ○ B. ○ C. ○ D. ○
5. A ○ B. ○ C. ○ D. ○

Girraween High School

Year 12 Mathematics Extension 2

Task 1, December 2024

General Instructions

- Reading Time - 5 minutes
- Working Time - 90 minutes
- Write using black pen only
- Calculators approved by NESA may be used
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total Marks: 66

Section I - 5 marks

- Attempt questions 1-5
- Allow about 10 minutes for this section

Section II - 61 marks

- Attempt questions 6-11
- Allow about 80 minutes for this section

Section I

5 marks
Attempt questions 1 - 5
Allow about 10 minutes for this section

Use the multiple choice answer sheet for Questions 1 to 5.

Question 1

A quadratic equation with zeros of $4 + i$ and $4 - i$ is

(A) $x^2 - 8x + 17 = 0$

(B) $x^2 + 8x + 17 = 0$

(C) $x^2 - 8x - 17 = 0$

(D) $x^2 + 8x - 17 = 0$

Question 2

The complex number w is a root of the equation $z^3 + 1 = 0$.

Which of the following is FALSE?

(A) $\overline{w}$ is also a root

(B) $w^2 + 1 - w = 0$

(C) $\frac{1}{w}$ is also a root

(D) $(w - 1)^3 = -1$

Question 3

$P(z)$ is a polynomial in z of degree 4 with real coefficients.

Which one of the following statements must be **false**?

(A) $P(z)$ has four real roots.

(B) $P(z)$ has two real roots and two non-real roots.

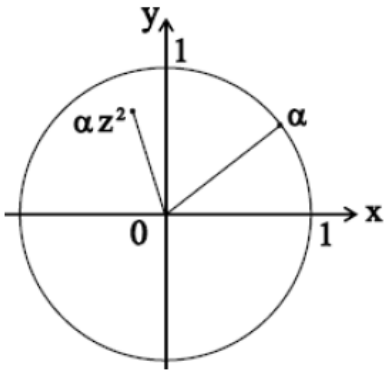
(C) $P(z)$ has one real root and three non-real roots.

(D) $P(z)$ has no real roots.

Examination continues next page…

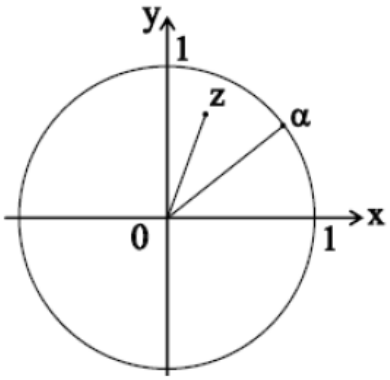
Question 4

The Argand diagram below shows the complex numbers α and αz^2.

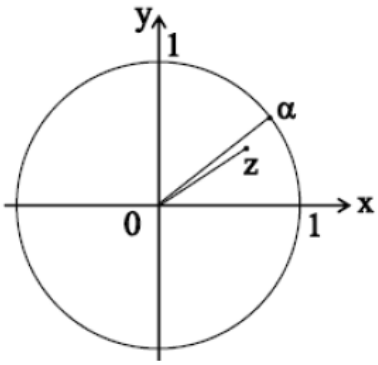


Which of the following best represents the positions of α and z?

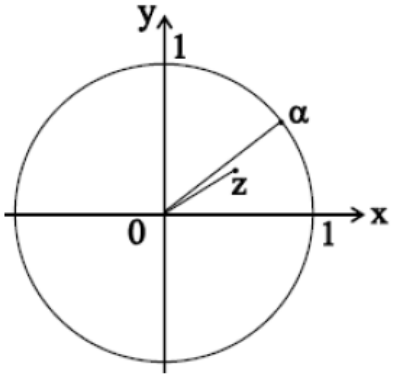
A)



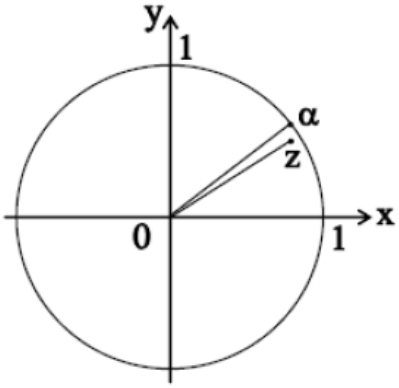
B)



C)



D)



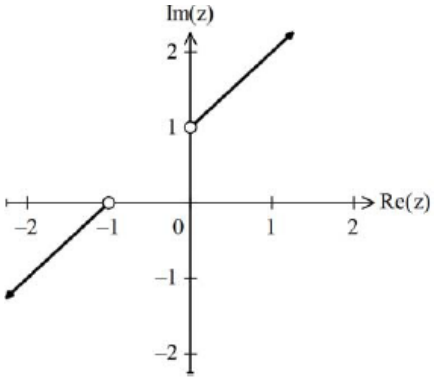
Examination continues next page…

Question 5

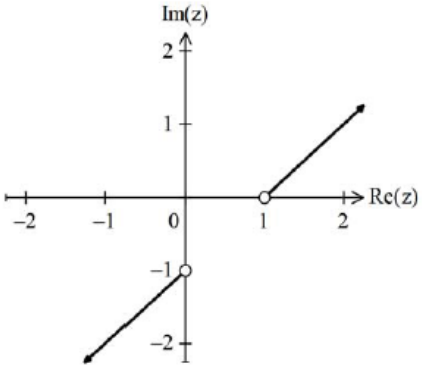
Which of the following Argand diagrams describes the locus defined by

$$\arg(z - i) = \arg(z + 1)?$$

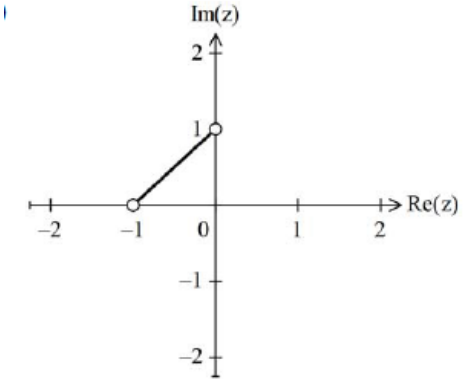
A)



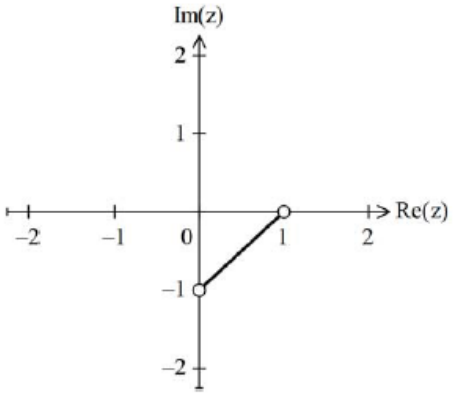
B)



C)



D)



End of Section 1

Examination continues next page…

Section II

61 marks
Attempt questions 6 - 11
Allow about 80 minutes for this section

Start each question on a new page.

In Questions 6 - 11, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 marks) Start a new answer page **Mark**

a) Given that $z_1 = 3 - i,\ z_2 = 2 + 5i,$ express in the form, $a + ib,$ where a and b are real

i) $(\overline{z_1})^2$ 1

ii) $\frac{z_1}{z_2}$ 1

iii) $\overline{\left|\frac{z_1}{z_2}\right|}$ 1

b) i) Express $1 - i\sqrt{3}$ in modulus-argument form. 1

ii) Hence evaluate $\left(1 - i\sqrt{3}\right)^6$. 1

c) If w is one of the complex roots of $z^3 - 1 = 0$, find the value of $\frac{1}{1+w} + \frac{1}{1+w^2}$. 2

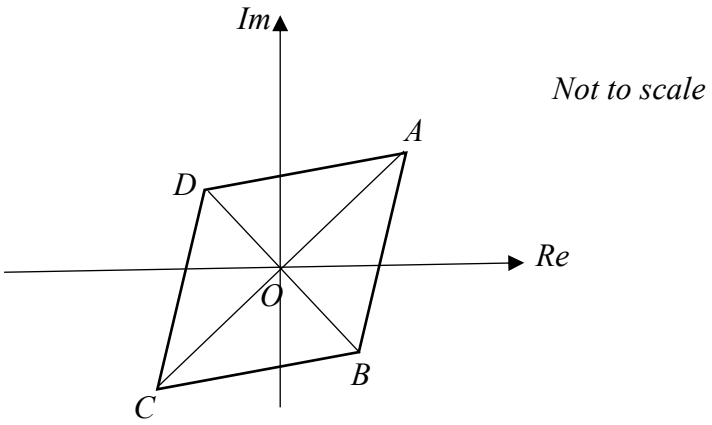
d) i) Find the complex square roots of $-18i$, expressing your answer in the form $a + bi$, where a and b are real numbers. 2

ii) Hence, solve the quadratic equation $z^2 - 3(1 + i)z + 9i = 0$ 2

Examination continues next page…

Question 7 (9 marks) Start a new answer page **Mark**

a) In the diagram below, *ABCD* is a rhombus whose diagonals meet at the origin, O.



If *A* represents the complex number $z_1 = 3 + 4i$ and *OD* = 2 units, find the complex number (in Cartesian form) represented by

i) D (let this point be z_2) 2

ii) C (let this point be z_3) 1

b) The complex number z and its conjugate $\bar{z}$ satisfies $(z - \bar{z})^2 + 8(z + \bar{z}) = 16$. 2

Find the Cartesian equation of the locus.

c) i) Find the complex sixth roots of unity in modulus-argument form. 2

ii) Sketch the roots on an Argand diagram. 2

Examination continues next page…

Question 8 (12 marks) Start a new answer page **Mark**

a) i) Given $\sqrt{3}+i=2cis\frac{\pi}{6}$. state $\sqrt{3}-i$ in modulus argument form. 1

ii) Hence, find the smallest positive integer *m*, such that $(\sqrt{3}+i)^m=\left(\sqrt{3}-i\right)^m$. 3

b) Consider the polynomial $P(x)=x^4-4x^3+11x^2-14x+12$. 4

Given $x=1-i\sqrt{2}$ is a root of $P(x)=0$, solve the equation $P(x)=0$.

c) Let $z^n=\cos n\theta+i\sin n\theta$

i) Show that $\sin n\theta=\frac{z^n-z^{-n}}{2i}$ and $\cos n\theta=\frac{z^n+z^{-n}}{2}$. 2

ii) Hence, or otherwise prove the identity 2

$$32\sin^4\theta\cos^2\theta=\cos 6\theta-2\cos 4\theta-cos2\theta+2.$$

Question 9 (10 marks) Start a new answer page **Mark**

a) (i) Draw a sketch of the locus of z given by the equation 2

$$\arg(z-1)-\arg(z+1)=\frac{\pi}{4}.$$

(ii) Find the centre of the circle on which this locus lies. 2

b) i) By considering $(cos\theta+isin\theta)^5$ using De Moivre's Theorem prove that 3

$$sin\,5\theta=sin\theta(5-20sin^2\theta+16\,sin^4\theta)$$

ii) By letting $\theta=\frac{\pi}{5}$, show that the exact value of $sin^2(\frac{\pi}{5})$ is $\frac{1}{8}(5-\sqrt{5})$. 3

Examination continues next page…

Question 10 (11 marks) Start a new answer page **Mark**

a) The complex number z satisfies both $|z-1| \leq |z-i|$ and $|z-2-2i| \leq 1$.

i) In an Argand diagram, indicate the region which contains the point P representing z. 3

ii) If P moves on the boundary of this region and $arg(z-1) = \frac{\pi}{4}$, find the value of z in the form $x+iy$ where x and y are real. 3

b) Let $z = e^{i\theta}$. Consider the expression $S_n = 1 + z + z^2 + \ldots + z^n$.

i) Show that $S_n = \frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}$ 1

ii) Show that $S_n = e^{\frac{in\theta}{2}} \times \frac{(\sin\frac{n+1}{2}\theta)}{\sin(\frac{\theta}{2})}$ 2

iii) Deduce that $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$ 2

Question 11 (8 marks) Start a new answer page **Mark**

Consider the polynomial $z^5 - i = 0$.

i) Find the roots of $z^5 = i$ 2

ii) Show that $(z-i)\left(z^2 - 2isin\frac{\pi}{10}z - 1\right)\left(z^2 + 2isin\frac{3\pi}{10}z - 1\right) = 0$ 4

iii) Given $z^5 - i = (z-i)(z^4 + iz^3 - z^2 - iz + 1)$, show that $sin\frac{\pi}{10}\ sin\frac{3\pi}{10} = \frac{1}{4}$. 2

End of examination.

Maths Ext 2 Solutions
2024-25 Task 1

Multichoice

Q1 $(x-4-i)(x-4+i)=0$

$[(x-4)-i][(x-4)+i]=0$

$(x-4)^2-i^2=0$

$(x-4)^2+1=0$

$x^2-8x+16+1=0$

$x^2-8x+17=0$ (A)

Q2 The roots of $z^3+1=0$

are ω, $-\omega$ and -1.

sum of roots $\omega+\frac{1}{\omega}-1=0$

which is equivalent to

$\omega^2+1-\omega=0$

(D) is false

Q3 A and D are possible.

However non real roots occur in conjugate pairs since coefficients are real.

∴ (C) is false.

Q4 Since $|z| < 1$

then $|z|^2 < |z| < 1$

and $\arg(\alpha z^2) = 2\arg(z) + \arg(\alpha)$

$\therefore$ (D)

Q5 (A)

Q6

a) i) $(\bar{z})^2 = (3+i)^2$ — very well done

$= 9 + 6i + i^2$

$= 8 + 6i$ (1)

ii) $\frac{z_1}{z_2} = \frac{3-i}{2+5i} \times \frac{2-5i}{2-5i}$

$= \frac{6 - 15i - 2i - 5}{4 - 25i^2}$ — very well done

$= \frac{1}{29} - \frac{17i}{29}$ (1)

iii) $\left|\overline{\frac{z_1}{z_2}}\right| = \left|\frac{1}{29} + \frac{17i}{29}\right|$

$= \sqrt{\left(\frac{1}{29}\right)^2 + \left(\frac{17}{29}\right)^2}$ — most students have done it well, some students forgot to find modulus.

$= \sqrt{\frac{10}{29}}$ (1)

OR $\left|\overline{\frac{z_1}{z_2}}\right| = \left|\frac{3-i}{2+5i}\right| = \frac{\sqrt{3^2+1^2}}{\sqrt{2^2+5^2}}$

$= \sqrt{\frac{10}{29}}$

Q6 b) i) $1 - \sqrt{3}i$

Now $|1 - \sqrt{3}i| = \sqrt{1^2 + \sqrt{3}^2} = 2$

$\arg(1 - \sqrt{3}i) = \tan^{-1}(-\sqrt{3})$ very well done

$= \frac{-\pi}{3}$

$\therefore 1 - \sqrt{3}i = 2\,\text{cis}\left(\frac{-\pi}{3}\right)$ (1)

ii) $(1 - \sqrt{3}i)^6 = \left[2 \cdot \text{cis}\left(\frac{-\pi}{3}\right)\right]^6$

$= 2^6\,\text{cis}\left(\frac{-6\pi}{3}\right)$ very well done

$= 64$ (1)

Most students have shown excellent understanding of Q6 except for some calculation mistakes.

c)

$$w^3 - 1 = 0$$

$$(w-1)(w^2 + w + 1) = 0$$

$\therefore w^3 = 1$

$w^2 + w + 1 = 0$ (since w is complex)] ①

$$\frac{1}{1+w} + \frac{1}{1+w^2} = \frac{1+w^2+1+w}{(1+w)(1+w^2)}$$

$$= \frac{1+w^2+1+w}{1+w^2+w+w^3}$$

$$= \frac{1+0}{0+1}$$ since $w^2 + w + 1 = 0$, $w^3 = 1$

$= 1$ ①

Students have shown good understanding of this concept. Most students used $w^3 = 1$ and $w^2 + w + 1 = 0$ but a few number of students found the roots and substituted them.

(Q6d) $z^2 = -18i$

let $z = a + ib$

$(a+ib)^2 = -18i$

$a^2 + 2abi + b^2 = -18i$

equate real & imaginary

$a^2 - b^2 = 0$ (1)

$2ab = -18$

$a = \frac{-9}{b}$ (2)

Subs (2) into (1)

$\left(\frac{-9}{b}\right)^2 - b^2 = 0$

$\frac{81}{b^2} - b^2 = 0$

$b^4 = 81$

$b = \pm 3$

$\therefore a = \frac{-9}{\pm 3} = \mp 3$

$\therefore$ The square roots are $3 - 3i$ and $-3 + 3i$

or $\pm 3 \mp 3i$.

Very well done. Almost all students have shown understanding of finding square roots

1

1

Q6 d(ii) $z^2 - 3(1+i)z + 9i = 0$

$$z = \frac{3(1+i) \pm \sqrt{9(1+i)^2 - 4(1)(9i)}}{2.1}$$

$$= \frac{3 + 3i \pm \sqrt{18i - 36i}}{2}$$

$$= \frac{3 + 3i \pm \sqrt{-18i}}{2} \quad (1)$$

$$= \frac{3 + 3i \pm (3 \mp 3i)}{2} \quad \text{from (i)}$$

$$= \frac{6}{2} \quad \text{or} \quad \frac{6i}{2}$$

$$= 3 \quad \text{or} \quad 3i \quad (1)$$

Very well done apart from a few calculation mistakes.

(Q9 a)(i) $|OA| = \sqrt{3^2+4^2} = 5$

z_2 is a $\frac{\pi}{2}$ rotation

from z_1, and $\frac{2}{5}$ the length

of z_1

Almost everyone could see the direction was $-3+4i$. However, many students didn't take into account the modulus of OD was 2 & didn't make $OD = \frac{2}{5}OA$.

$\therefore z_2 = z_1 \times i \times \frac{2}{5}$

$= (3+4i) \times i \times \frac{2}{5}$

$z_2 = -\frac{8}{5} + \frac{6}{5}i$

① multiplying z by i

① multiplying z_1 by $\frac{2}{5}$

(ii) z_3 is a rotation of π from z_1

$\therefore z_3 = -1 \times (3+4i)$

$= -3-4i$ ①

Very simple.
Almost everyone was correct.

Q7b)

$$(z-\bar{z})^2 + 8(z+\bar{z}) = 16$$

let $z = x+iy$

$$[x+iy-(x-iy)]^2 + 8[x+iy-(x-iy)] = 16 \quad \text{①}$$

Now $z+\bar{z} = 2Re(z) = 2x$

$z-\bar{z} = 2i\,Im(z) = 2iy$

$$(2iy)^2 + 8(2x) = 16$$

$$-4y^2 = 16-16x$$

$$y^2 = 4(x-1) \quad \text{①}$$

Most students realised to let $z = x+iy$ and also realised $z-\bar{z} = 2iy$ & $z+\bar{z} = 2x$.
Some students forgot that $(2iy)^2 = -4y^2$; NOT $4y^2$.

Q7 c)

i) $z^6 = 1$

let $z = \text{cis}\,\theta$

$z^6 = (\text{cis}\,\theta)^6 = \text{cis}\,6\theta$

Now $1 = \text{cis}\,0$

equating arguments

$6\theta = 0 + 2k\pi$

$\theta = \frac{k\pi}{3}$ for $k = 0, \pm 1, \pm 2, 3$

(1) proper use

Almost everyone got (i) right.

$z_1 = \text{cis}\,0 = 1$

$z_2 = \text{cis}\,\frac{\pi}{3}$

$z_3 = \text{cis}\,\frac{-\pi}{3}$

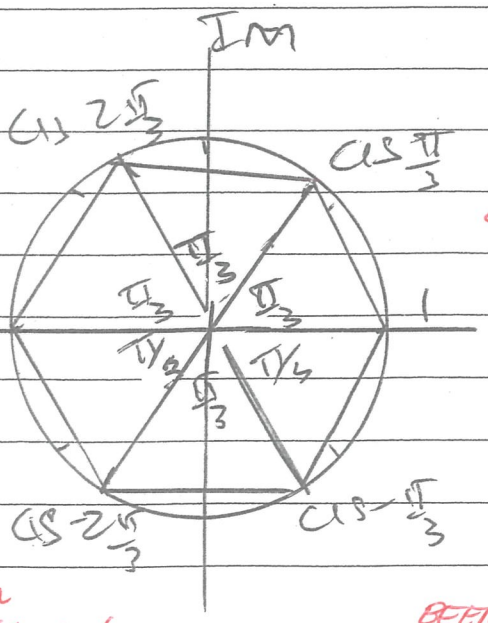
$z_4 = \text{cis}\,\frac{2\pi}{3}$

$z_5 = \text{cis}\,-\frac{2\pi}{3}$

$z_6 = \text{cis}\,\pi = -1$

(1)

(ii)



A simple question done with a LOT of careless ommissions.

Students often didn't either (1) show the angles were $\frac{\pi}{3}$

(1) shape or

(1) labels (2) show the moduli were all 1

[which SHOULD HAVE BEEN DONE BY PUTTING THEM ALL ON A UNIT CIRCLE!

Marking note:

(1) If they labelled cis $\frac{\pi}{3}$ etc. I counted that as angle labels.

(2) if they labelled all of the moduli as 1 I didn't penalise them for not having the unit circle. [This was very rare].

(Q 8 a) i) $z = 2\operatorname{cis}\frac{\pi}{6}$

$\therefore \bar{z} = 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$ (1)

well done

(ii) $(\sqrt{3}+i)^n = (\sqrt{3}-i)^n$

$\left(2\operatorname{cis}\frac{\pi}{6}\right)^n = \left(2\operatorname{cis}-\frac{\pi}{6}\right)^n$

$2^n \operatorname{cis}\frac{n\pi}{6} = 2^n \operatorname{cis}-\frac{n\pi}{6}$ (1)

by De Moivre's

$\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6} = \cos\left(-\frac{n\pi}{6}\right) + i\sin\left(\frac{-n\pi}{6}\right)$

Now, $\cos\frac{n\pi}{6} = \cos\left(-\frac{n\pi}{6}\right)$

and $\sin\left(-\frac{n\pi}{6}\right) = -\sin\frac{n\pi}{6}$

$\therefore 2i\sin\frac{n\pi}{6} = 0$ (1)

$\frac{n\pi}{6} = \pi$

$\therefore$ smallest positive integer n is $\underline{6}$ (1) well done

Q 8 b)

Since the coefficients of the polynomial are real * must mention

$z = 1 + \sqrt{2}i$ is also a root. (1)

$[z-((1+\sqrt{2}i)][z+(1+\sqrt{2}i)]$

Now $z^2 - 2\,Re(z) + |z|^2$

$= z^2 - 2z + (1^2 + \sqrt{2}^2)$

$= z^2 - 2z + 3$ is a factor (1)

$$z^2-2z+3 \;\overline{)\; z^4 - 4z^3 + 11z^2 - 14z + 12}$$

$$z^4 - 2z^3 + 3z^2$$

$$-2z^3 + 8z^2 - 14z$$

$$-2z^3 + 4z^2 - 6z$$

$$4z^2 - 8z + 12$$

$$4z^2 - 8z + 12$$

$$0$$

(1)

$z^2 - 2z + 4 = 0$

$z = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$ well done

$= \dfrac{2 \pm 2\sqrt{3}i}{2} = 1 \pm \sqrt{3}i$ (1)

∴ solutions are $x = 1 \pm \sqrt{2}i$, $x = 1 \pm \sqrt{3}i$

Q 8 c) i)

Given $z^n = \cos n\theta + i \sin n\theta$ (1)

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

since $\cos(-n\theta) = \cos n\theta$

and $\sin(-n\theta) = -\sin n\theta$

$\therefore z^{-n} = \cos n\theta - i \sin n\theta$ (2)

(1) + (2):

$z^n + z^{-n} = 2\cos n\theta$

$\therefore \cos n\theta = \frac{1}{2}(z^n + z^{-n})$

(1) − (2):

$z^n - z^{-n} = 2i \sin n\theta$

$\therefore \sin n\theta = \frac{1}{2i}(z^n - z^{-n})$

well done.
students penalised if
$\sin(-n\theta) = -\sin(n\theta)$
not shown.

Q 8 (c) (ii)

RHS $= 32\sin^4\theta\cos^2\theta$

$= 32\sin^4\theta(1-\sin^2\theta)$

$= 32\sin^4\theta - 32\sin^6\theta$

(1) $= 32\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^4 - 32\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^6$

$= 2\left(z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}\right)$

(1) $+ \frac{1}{2}\left(z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}\right)$

$= \frac{1}{2}\left(z^6 + \frac{1}{z^6}\right) - \left(z^4 + \frac{1}{z^4}\right)$

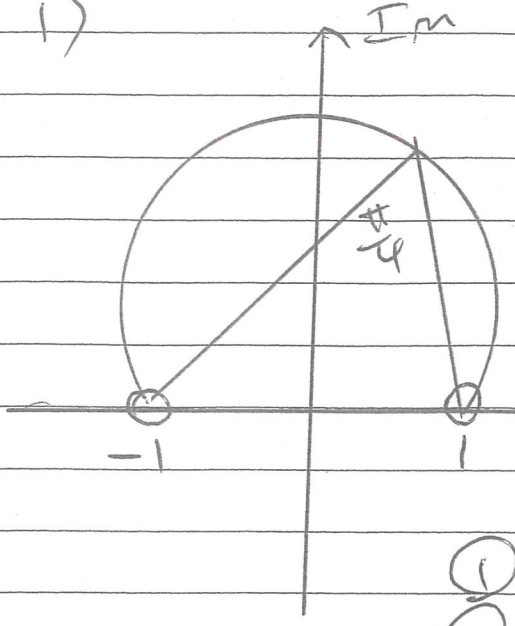
$- 2\left(z^2 + \frac{1}{z^2}\right) + 2$

$= \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

$=$ LHS

poorly done

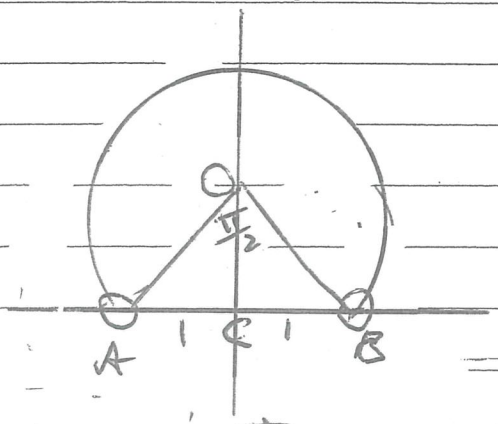
Q9 a) i)



some few students forgot the arc and some students forgot the open circles

① arc
① correct end pts

ii) centre lies on the vertical axis.

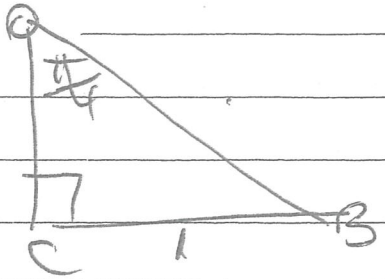


① reasoning
① centre

$\angle AOB = \frac{\pi}{2}$

Angle at the centre is twice the angle at the circumference.

$\triangle AOB$ is also isosceles



$\therefore \tan\frac{\pi}{4} = \frac{1}{OC}$

$\therefore OC = 1$

$\therefore$ centre $(0, 1)$

very well done
Students to write C(0, 1) however not penalised

9(5) i)

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

by De Moivre's Theorem

1 5 10 10 5 1

$(\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5\cos^4\theta\, i\sin\theta + 10\cos^3\theta\, i^2\sin^2\theta$

$+ 10\cos^2\theta\, i^3\sin^3\theta + 5\cos\theta\, i^4\sin^4\theta$

$+ i^5\sin^5\theta$

(1) Expanding

$= \cos^5\theta + 5\cos^4\theta\, i\sin\theta - 10\cos^3\theta\sin^2\theta$

$- 5\cos\theta\, i\sin^3\theta + i\sin^5\theta$

equating imaginary parts

$\sin 5\theta = 5\cos^4\theta\sin\theta$

(1) $- 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= \sin\theta(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta)$

apply $\cos^2\theta = 1 - \sin^2\theta$

$= \sin\theta[5(1-\sin^2\theta)^2 - 10(1-\sin^2\theta)\sin^2\theta + \sin^4\theta$

$= \sin\theta[5(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^2\theta + 10\sin^4\theta + \sin^4\theta$

(b) $\sin^2\frac{\pi}{5} = \frac{5+\sqrt{5}}{8}$ or $\frac{5-\sqrt{5}}{8}$

Now $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\sin^2\frac{\pi}{4} = \frac{1}{2}$

$\sin^2\frac{\pi}{5} < \sin^2\frac{\pi}{4} < \frac{1}{2}$ } justification

$\therefore \sin^2\frac{\pi}{5} = \frac{5-\sqrt{5}}{8}$ (1)

Very well done
apart from justification

Most students who lost a mark as they did not give a convincing justification.

Q10

a) let $z = x + iy$

$|z-1| \leq |z-i|$

$|x+iy-1| \leq |x+iy-i|$

$|x-1+iy| \leq |x+i(y-1)|$

$\sqrt{(x-1)^2+y^2} \leq \sqrt{x^2+(y-1)^2}$

$x^2-2x+1 \leq x^2+y^2-2y+1$

$-2x \leq -2y$

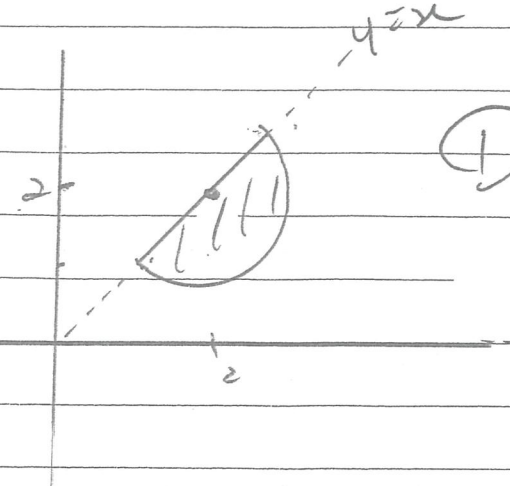
$y \leq x$ (1) or correct graph

The vast majority of students did (10)(a)(i) well.

$|z-2-2i| \leq 1$

$|z-(2+2i)| \leq 1$

This is a circle with centre $(2,2)$ and radius 1 (1) or correct graph

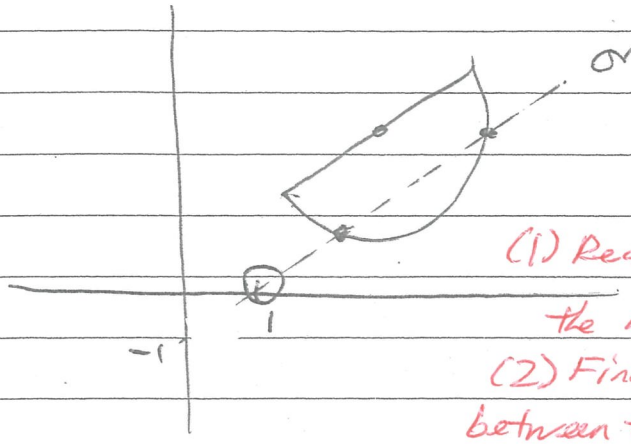


(1) correct region

(no penalty if $y=x$ is all solid)

ii) If $\arg(z-1) = \frac{\pi}{4}$

then z lies on the line $y = x - 1$ (1)



Many students had trouble
(1) Realising $\arg(z-1) = \frac{\pi}{4}$ is the line $y = x-1, x > 1$
(2) Finding BOTH points of intersection between the line $y = x-1$ & the circle.

Solve $y = x - 1$ (1)
& $(x-2)^2 + (y-2)^2 = 1$ (2)
simultaneously

Subs (1) into (2)

$(x-2)^2 + (x-1-2)^2 = 1$ (1) progress

$x^2 - 4x + 4 + x^2 - 6x + 4 = 1$

$2x^2 - 10x + 12 = 0$

$x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x = 2$ $\quad$ $x = 3$

$y = 1$ $\quad$ $y = 2$

$\therefore \quad z = 2 + i \quad$ or $\quad 3 + 2i$ (1)

Q10 b)

i) $S_n = 1 + e^{i\theta} + e^{i2\theta} + e^{i3\theta} + \ldots + e^{in\theta}$

This is a GP with

$a = 1 \quad r = e^{i\theta} \quad n = n+1$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{1\left(e^{i\theta(n+1)} - 1\right)}{e^{i\theta} - 1}$$

$$= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$$

①

Almost everyone realised it was the sum of a GP but I needed some WORKING → either the sum of a GP formula or $a =$ $r =$ and $n =$

Otherwise I could NOT award the mark.

ii) Take out $e^{i\left(\frac{n+1}{2}\right)\theta}$ and $e^{i\frac{\theta}{2}}$ as common factors

$$S_n = \frac{e^{i\left(\frac{n+1}{2}\right)\theta}\left(e^{i\left(\frac{n+1}{2}\right)\theta} - e^{-i\left(\frac{n+1}{2}\right)\theta}\right)}{e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)}$$ ①

Very poorly done. Some students went to De Moivre (only 1 succeeded this way)

Now $z + \frac{1}{z} = 2i\sin n\theta$

$$= \frac{e^{i\left(\frac{n+1}{2}\right)\theta}\left[2i\sin\left(\frac{n+1}{2}\right)\theta\right]}{e^{i\frac{\theta}{2}}\left(2i\sin\frac{\theta}{2}\right)}$$ ①

$$\therefore S_n = \frac{e^{i\left(\frac{n+1}{2}\right)\theta} \sin\left(\frac{n+1}{2}\right)\theta}{e^{i\frac{\theta}{2}} \sin\frac{\theta}{2}}$$

(ii) $\mathrm{Im}\left(1 + z + z^2 + \ldots + z^n\right)$

$= \sin\theta + \sin 2\theta + \ldots \sin n\theta$ (1)

About 15 or 20 students realised to equate imaginary parts & they all succeeded with this question.

from (ii)

$$S_n = \frac{\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right) \times \sin\left(\frac{n+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

expand

$$(1) = \frac{\cos\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}} + i\frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\therefore \mathrm{Im}(S_n) = \frac{\sin\frac{n\theta}{2}\sin\left(\frac{n+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$$\therefore \sin\theta + \sin 2\theta + \ldots + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\left(\frac{n+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

Q11 i) let $z^5 = r^5 \operatorname{cis} 5\theta$

$$i = 1 \operatorname{cis} \frac{\pi}{2}$$

$r^5 = 1$ $\quad 5\theta = \frac{\pi}{2} + 2k\pi$

$\therefore r = 1$ $\quad \theta = \frac{\pi}{10} + \frac{2k\pi}{5}$

$$= \frac{\pi + 4k\pi}{10}$$ (1)

where $k = 0, \pm 1, \pm 2$

$z_1 = \operatorname{cis} \frac{\pi}{10}$

$z_2 = \operatorname{cis} \frac{\pi}{2} = i$

$z_3 = \operatorname{cis} \frac{9\pi}{10}$

$z_4 = \operatorname{cis} \left(-\frac{3\pi}{10}\right)$

$z_5 = \operatorname{cis} \left(-\frac{7\pi}{10}\right)$ (1)

Well done.

ii) $z^5 - i = 0$ can be expressed as

$$(z-i)\left(z-\text{cis}\frac{\pi}{10}\right)\left(z-\text{cis}\frac{9\pi}{10}\right)\left(z-\text{cis}\frac{-3\pi}{10}\right)\left(z-\text{cis}\frac{-7\pi}{10}\right) \quad (1)$$

Now, $\alpha = \text{cis}\frac{\pi}{10}$ and $-\bar{\alpha} = \text{cis}\frac{9\pi}{10}$ (1)

notice:

many students did not realise this

$\beta = \text{cis}\frac{-3\pi}{10}$ and $-\bar{\beta} = \text{cis}\frac{-7\pi}{10}$

$$\therefore (z-i)(z-\alpha)(z+\bar{\alpha})(z-\beta)(z+\bar{\beta}) = 0$$

$$(z-i)\left[z^2 + (\bar{\alpha}-\alpha)z - \alpha\bar{\alpha}\right]$$

$$\left[z^2 + (\bar{\beta}-\beta)z - \beta\bar{\beta}\right] = 0$$

Now $\alpha\bar{\alpha} = 1$ $\beta\bar{\beta} = 1$ (1)

and

$$\bar{\alpha} - \alpha = \cos\frac{\pi}{10} - i\sin\frac{\pi}{10} - \cos\frac{\pi}{10} - i\sin\frac{\pi}{10}$$

$$= -2i\sin\frac{\pi}{10}$$

The more successful students wrote out the expansion. (1)

Similarly

$$\bar{\beta} - \beta = -2i\sin\frac{-3\pi}{10}$$

$$= 2i\sin\frac{3\pi}{10}$$

$\therefore (z-i)\left(z^2 - 2\sin\frac{\pi}{10}z - 1\right)$

$\left(z^2 + 2\sin\frac{3\pi}{10}z - 1\right)$

Poorly done as required
All steps must be shown to obtain full marks.

(iii) $(z-i)(z^4 + iz^3 - z^2 - iz + 1)$

$= (z-i)\left(z^2 - 2i\sin\frac{\pi}{10}z - 1\right)$

$\left(z^2 + 2i\sin\frac{3\pi}{10}z - 1\right)$ ①

equate coefficients of z^2

$-1 = -2i\sin\frac{\pi}{10} \cdot 2i\sin\frac{3\pi}{10} - 1 - 1$ ①

$1 = -4i^2 \sin\frac{\pi}{10}\sin\frac{3\pi}{10}$

$\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$

Generally well done
Please show working explicitly!